Signal Analysis & Communication ECE 355

Ch. 4.4 & 4.5: The Convolution & Multiplication Properties of CTFT

Lecture 22

30-10-2023

- As we have seen in the properties of CTFS for Periodic signals that multiplication in time-domain is convolution in freq. domain & vice versa, similar is the case for CTFT for Aperiodic signals, that is,

" Convolution in time-domain for CT Aperiodic signals is multiplication in freq.-domain & vice versa."

- We already have stated & used these properties in solving LCCDE

- Here, we will first derive the properties (Convolution & multiplication) & see their applications on practical problems (Filtering & Communication sys.)

Ch. 4.4 The Convolution Property

$$x(t) * h(t) \xleftrightarrow{\mathcal{F}} X(j\omega)\, H(j\omega)$$

Recall: $e^{st} \longrightarrow \boxed{h(t)} \longrightarrow H(s)\, e^{st} = y(t)$

& $H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$

using $y(t) = x(t) * h(t)$
$= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$
$= e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau}_{H(s)}$

- Then we applied this to Periodic sig. (expressed as CTFS, that is, sum of harmonically related complex sinusoids).

- Lets extend this idea for applying to Aperiodic sig. (expressed as CTFT, that is, integral of complex sinusoids)

- Therefore, if

$$y(t) = x(t) * h(t)$$

$$= \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \right) * h(t)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{X(j\omega) \left[H(j\omega)\right.}_{Y(j\omega)]} \left. e^{j\omega t} \right] d\omega$$

$$x(t) * h(t) \longleftrightarrow X(j\omega)\, H(j\omega)$$

$$\Rightarrow \quad x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\Updownarrow$$

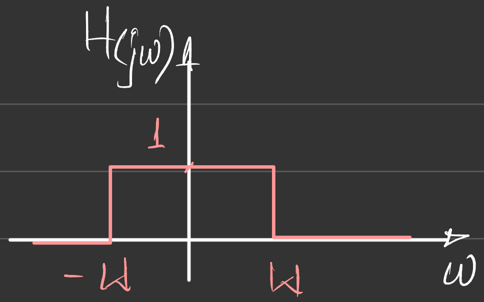
$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

Freq. Response

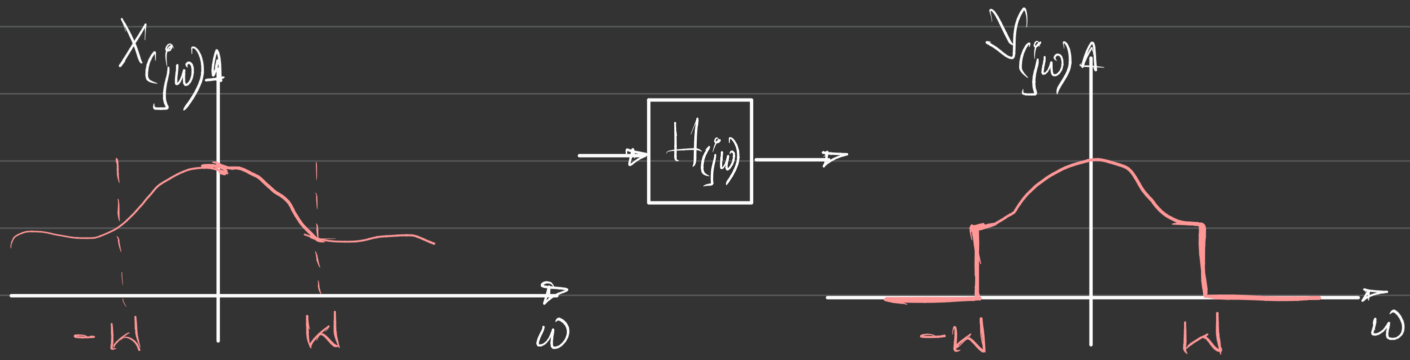
- As told previously, the freq. response plays an important role in the analysis of LTI sys.

- For example, in freq. selective filtering we may want $H(j\omega) \simeq 1$ over one range of frequencies, so that the freq. components in this band experience little or no attenuation, while over another range of frequencies we may want to have $H(j\omega) \simeq 0$, so that the components in this range are eliminated.

- Example



"Ideal Low-pass Filter"



- As we had seen for the two cascaded systems, the impulse response is:

$$x_{(t)} \longrightarrow \underbrace{\boxed{h_1(t)} \longrightarrow \boxed{h_2(t)}}_{h_1(t) * h_2(t) = h_2(t) * h_1(t)} \longrightarrow y(t)$$

- The Freq. Response is given as:

$$X_{(j\omega)} \longrightarrow \underbrace{\boxed{H_1(j\omega)} \longrightarrow \boxed{H_2(j\omega)}}_{H_1(j\omega)\, H_2(j\omega) = H_2(j\omega)\, H_1(j\omega)} \longrightarrow Y_{(j\omega)}$$

$$X_{(j\omega)} \longrightarrow \boxed{H_1(j\omega)\; H_2(j\omega)} \longrightarrow Y_{(j\omega)}$$

- Example

- Recall: $\dfrac{d}{dt} x_{(t)} \longleftrightarrow j\omega X_{(j\omega)}$

$y(t) \longrightarrow$

$$X_{(j\omega)} \rightarrow \boxed{j\omega} \rightarrow Y_{(j\omega)}$$

- And if 'n' differential

$$\underbrace{\rightarrow \boxed{j\omega} \rightarrow \boxed{j\omega} \cdots \rightarrow \boxed{j\omega}}_{n \text{ of them}} \rightarrow$$

$$\rightarrow \boxed{(j\omega)^n} \rightarrow$$

- So, $\frac{d^n}{dt^n} x_{(t)} \longleftrightarrow j^n \omega^n X_{(j\omega)}$

*seen this in solving LCCDE via CTFT (Lec. 21)

Ch. 4.5 The Multiplication Property

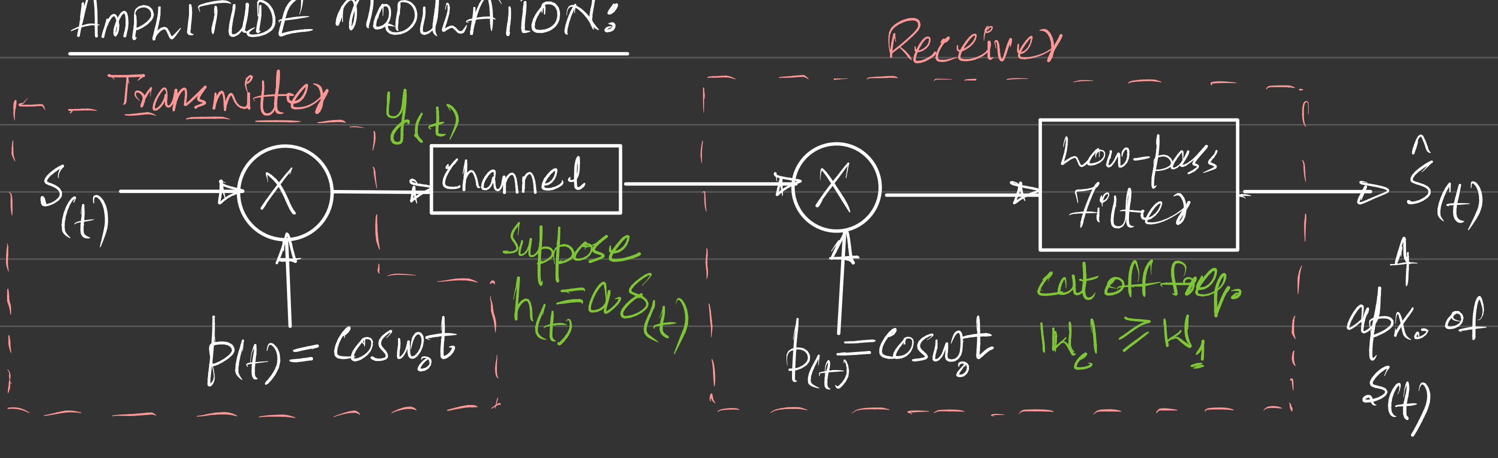
$$x_{1(t)} x_{2(t)} \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} (X_1 * X_2)_{(j\omega)}$$

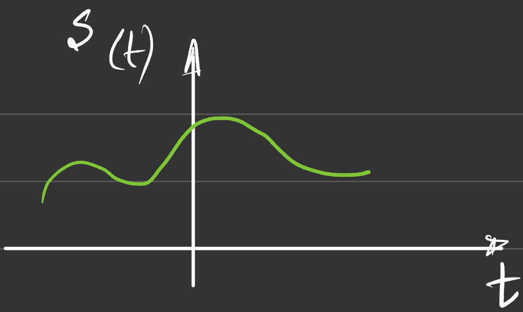
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_{1(j\theta)} X_{2(j(\omega-\theta))} d\theta$$

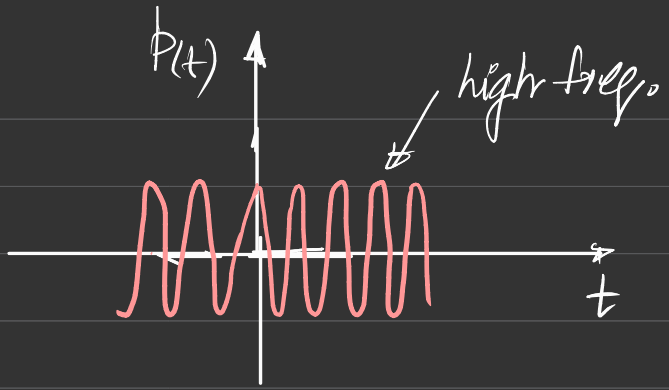
- Proof: via Duality or Direct.

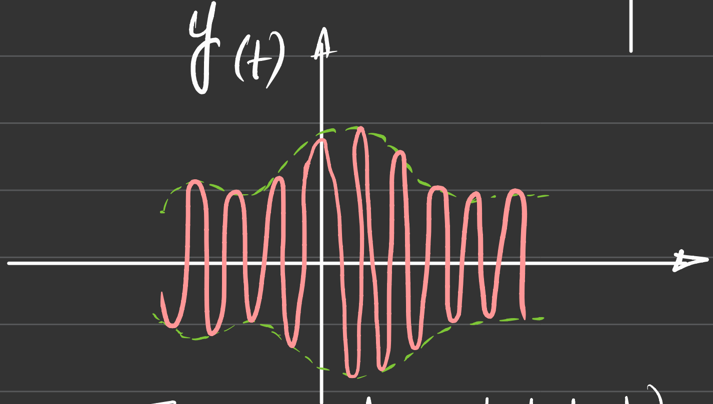
- Example

AMPLITUDE MODULATION:



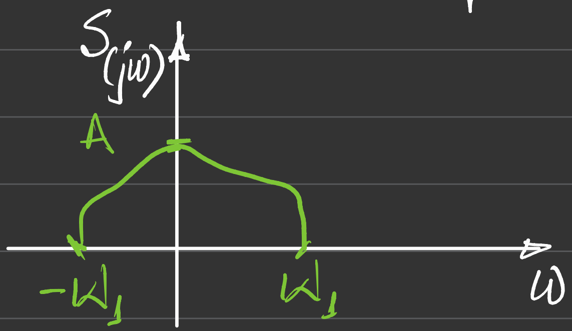






$y_{(t)}$: Transmitted (modulated) sig. which is now at high freq. to be suitable for wireless comm. medium (channel)

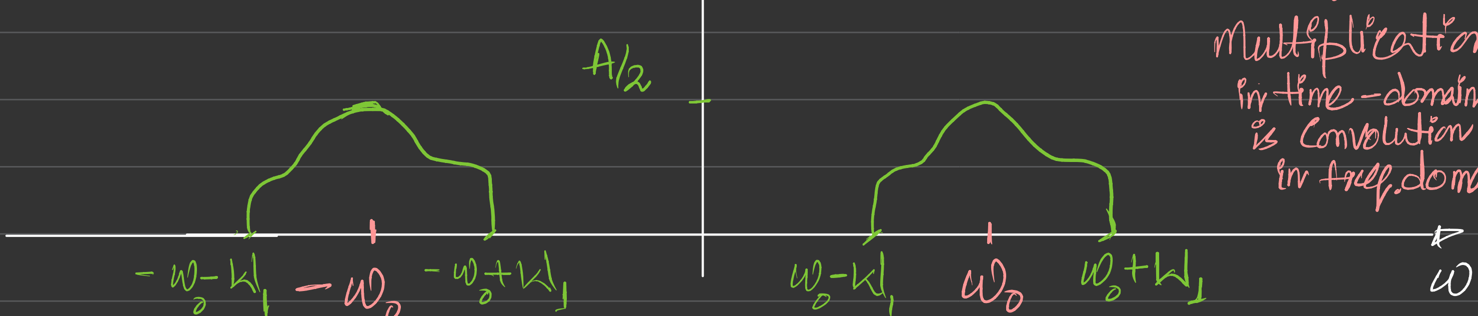
Let's look at the freq-domain now!



$P_{(j\omega)} = \pi(\delta_{(\omega-\omega_0)} + \delta_{(\omega+\omega_0)})$

$p_{(t)} = \cos\omega_0 t$
$= \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$
$P_{(j\omega)} = \frac{1}{2}[2\pi\delta_{(\omega-\omega_0)} + 2\pi\delta_{(\omega+\omega_0)}]$
(using freq. shift property)

$Y_{(j\omega)} = \frac{1}{2\pi}(S_{(j\omega)} * P_{(j\omega)})$



$y_{(t)} = x_{(t)}\, p_{(t)}$
multiplication in time-domain is convolution in freq. domain

* Remember convolution with an impulse is an identity system.

$$Y_{(j\omega)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_{(j\lambda)} X_{(j(\omega-\lambda))}\, d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi\,(\underbrace{\delta_{(\lambda-\omega_0)}}_{\lambda=\omega_c} + \underbrace{\delta_{(\lambda+\omega_0)}}_{\lambda=-\omega_c})\, X_{(j(\omega-\lambda))}\, d\lambda$$

$$= \frac{1}{2}\left[X(j(\omega-\omega_0)) + X(j(\omega+\omega_0))\right] \text{ — (A)}$$

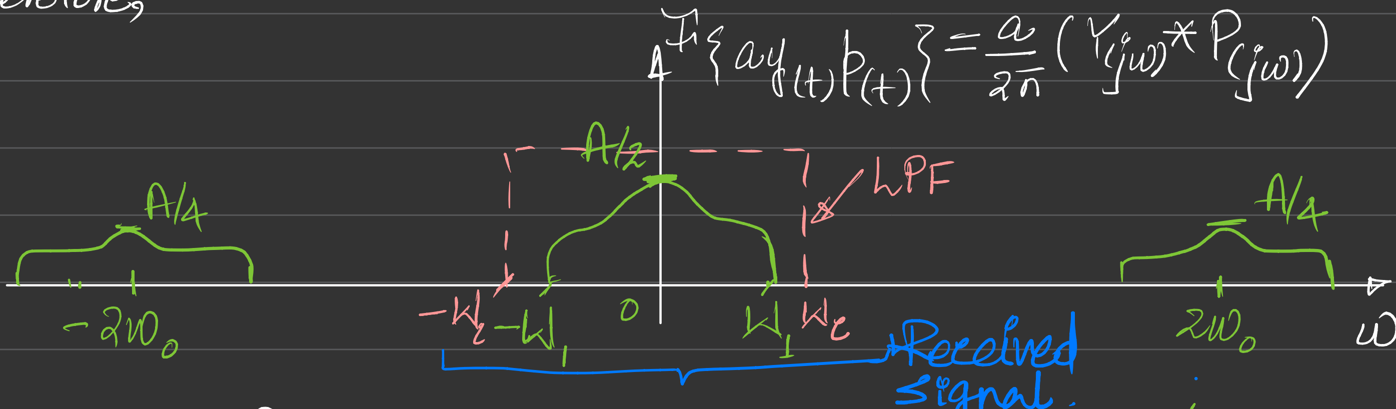
At the Receiver:

$\hat{s}_{(t)}$ is a low-pass filtered sig. of $\underbrace{[y_{(t)} * \overbrace{a\delta_{(t)}}^{h_{(t)} \text{ of channel}}]}\, p_{(t)}$

$$\xleftrightarrow{\mathcal{F}} \quad = \underbrace{a y_{(t)} p_{(t)}}$$

$\Rightarrow \hat{S}_{(j\omega)}$ is low-pass filtered version of $\frac{a}{2\pi}\left(Y_{(j\omega)} * P_{(j\omega)}\right)$

- Therefore,



* Convolving (A) with $\pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$ ($\mathcal{F}\{\cos\omega_0 t\}$)

$$\frac{a}{2\cancel{\pi}}\left[\frac{\cancel{\pi}}{\cancel{2}}\left(\underbrace{\cancel{2}X_{(j\omega)}}_{\text{Rcv sig.}} + \cancel{X(j(\omega-2\omega_0)) + X(j(\omega+2\omega_0))}\right)\right]$$

Filtered out.

* Same as transmitted $s_{(t)}$, only scaled $\therefore$ represented as $\hat{s}_{(t)}$

Remember:

$$x_{(t)} * \delta_{(t)} = x_{(t)}$$

$$x_{(t)} * \delta_{(t-\tau)} = x_{(t-\tau)}$$

$$X_{(j\omega)} * \delta_{(j\omega)} = X_{(j\omega)}$$

$$X_{(j\omega)} * \delta_{(\omega-\omega_0)} = X_{(j(\omega-\omega_0))}$$